ANALITICAL STUDY OF THE INFLUENCE OF BENDING AND SHEAR STIFFNESS AND ROTATIONAL INERTIA IN THE DYNAMIC BEHAVIOR OF BEAMS, PART I: THEORY

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Abstract. This paper shows the analytical development of the dynamic motion equation of beams, based on Bernoulli and Timoshenko's theories, using the Direct Method approach. The influence of the effects of bending and rotational inertia on the behaviour of beams were taken into account based on Bernoulli's theory, while the influence of bending and shear stiffness, and rotational inertia were based on Timoshenko's theory. This study presents the analytical development of the dynamic motion equation of undamped beams, for different boundary conditions. From these equilibrium equations, the natural frequencies for each case analysed were determined. In this context, the natural frequencies were obtained by means of the transcendental functions for both, Bernoulli and Timoshenko's beam theories. In order to solve such equations, the Newton-Raphson method is employed..

1 INTRODUCTION

This work presents the analytical development of the dynamic motion equation of undamped beams for four different boundary conditions: pinned-pinned, fixed-free, fixed-pinned and fixed-fixed. Two theories are considered: Bernoulli's theory is used to analysis the effects of bending and rotational inertia on the behavior of beams, while Timoshenko's theory is used to study the influence of bending and shear stiffness, and rotational inertia. The second section of this paper shows, from the Direct Method, the obtainment of the dynamic equation of the beam's motion. Next in the third section, the analytical expressions of the vertical displacement of the beam, referring to the four boundary conditions studied, are presented for Bernoulli's theory. The fourth section presents the hypothesis made by Timoshenko's theory and shows the analytical expressions of the vertical displacement and of the rotation of the beam for the four boundary conditions.

2 THE OBTAINMENT OF THE DYNAMIC EQUATION OF THE BEAM

Figure 1 shows the free body diagram of an infinitesimal beam element oriented along the *x* axis at a time-step *t*.

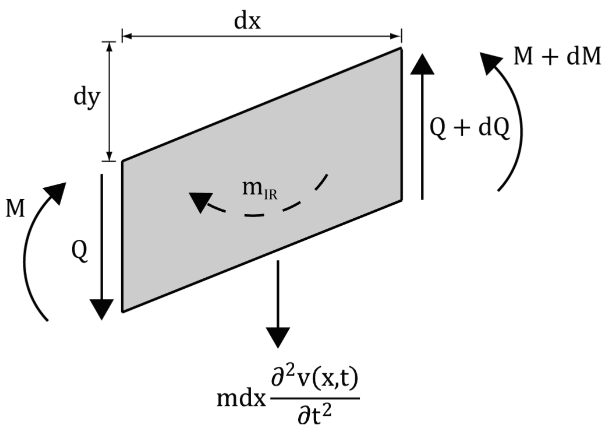

Figure 1: Infinitesimal element considering rotational inertia (Adapted to Sousa et al, 2011)

In Fig. 1, m is the mass per unit length,   is the shear force,   is the bending moment, dx is the infinitesimal element's length, dv is the vertical displacement increment and,   is the rotational inertia, which can be expressed as:

$$m_{IR}(x,t) = \left(\frac{mI}{A}\right)\frac{\partial^2\theta(x,t)}{\partial t^2} \tag{1}$$

in which A is the cross section area, I is the moment of inertia and θ is rotation of the cross section.

From the Direct Method and doing the force balance in the vertical direction, one can write:

$$\frac{\partial Q(x,t)}{\partial x} = m\frac{\partial^2 v(x,t)}{\partial t^2} \tag{2}$$

The moment balance equation can be expressed by:

$$\frac{\partial M(x,t)}{\partial x} = -Q(x,t) + m_{IR}(x,t) \tag{3}$$

The constitutive relation is given by:

$$M(x,t) = EI\frac{\partial \theta(x,t)}{\partial x} \tag{4}$$

and

$$Q(x,t) = GA\kappa\,\gamma_{xy}(x,t) \tag{5}$$

in which $\theta(x,t)$ represents the rotation of a generic cross section, $\gamma_{xy}(x,t)$ is the shear distortion and is the area correction.

By combining the equations (2), (3), (4) and (5), regardless of the beam theory adopted, a fourth order differential equation is obtained in which the vertical displacement, $v(x,t)$, is unknown. Next, using a differential operator *D*, (*D* = d/d*t*), and the Method of Separation of Variables (MSV) considering that $v(x,t)$ can be decoupled into a spatial function $w(x)$ and a function of time $q(t)$, the following expression is obtained:

$$\left(D^4 + 0D^3 + B(\omega, E, I, G, A)D^2 + 0D + (-1)R(\omega, E, I, G, A)\right)w(x) = 0 \tag{6}$$

in which ω are the natural frequencies, and $B(\omega, E, I, G, A)$ and $R(\omega, E, I, G, A)$ depends on parameters ω, *E*, *I*, *G* and *A*. For simplicity of notation, it will be considered, further on, that $B(\omega, E, I, G, A) = B$ and $R(\omega, E, I, G, A) = R$. The obtainment of Eq. (6) is done with more details by Sousa *et al* (2011) that showed the decoupling process of the functions $w(x)$ and $q(t)$. In this decoupling process, a harmonic solution of the type "$q(t) = \sin(\omega t + \phi)$"is assumed. In this harmonic function, ϕ is the phase angle. Sousa *et al* (2011) disregarded the trivial solution of Eq. (6), and presented Eq. (7):

$$\left(D^4 + 0D^3 + B(\omega, E, I, G, A)D^2 + 0D + (-1)R(\omega, E, I, G, A)\right) = 0 \tag{7}$$

Equation (7) can be interpreted as a fourth degree polynomial in the variable *D*. According to Boas (1966), an algebraic expression can be written as a function of their roots, in which each coefficient represents a different combination of the roots of algebraic expression. Thus, considering a polynomial of fourth degree complete with roots *a*, *b*, *c* and *d*, one can write that:

$$\begin{aligned}(x-a)(x-b)(x-c)(x-d) &= x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 \\ &- (bcd+acd+abd+abc)x + abcd = x^4 + Ax^3 + Bx^2 + Cx + R\end{aligned} \tag{8}$$

Comparing the fourth degree polynomial represented by Eq. (7) with the polynomial shown in Eq. (8), one obtains the following system of equations:

$$\begin{cases} a+b+c+d=0 \\ ab+ad+ac+bc+bd+cd=B \\ abc+bcd+acd+abd=0 \\ abcd=R \end{cases} \tag{9}$$

Sousa et al (2011) and Ervik et al (1986) showed that the values of *a*, *b*, *c*, and *d*; are respectively:

$$a = +\delta,\ b = -\delta,\ c = +i\beta \text{ and } d = -i\beta \tag{10}$$

in which:

$$\delta = \sqrt{\frac{-B + \sqrt{B^2 + 4R}}{2}},\ \beta = \sqrt{\frac{B + \sqrt{B^2 + 4R}}{2}} \text{ and } i = \sqrt{-1} \tag{11}$$

After determining the roots of the polynomial, it is possible to obtain the solution of the spatial function "$w(x)$" (Boas, 1966):

$$w(x) = C_1 e^{+x\delta} + C_2 e^{-x\delta} + C_3 e^{+ix\beta} + C_4 e^{-ix\beta} \tag{12}$$

Equation (12) can be written in another way:

$$w(x) = C_1 \cosh(x\delta) + C_2 \sinh(x\delta) + C_3 \cos(x\beta) + C_4 \sin(x\beta) \tag{13}$$

Equation (13) is the general solution of the spatial function of the dynamic displacement of beams. Sousa et al (2011) showed that this same equation is also the solution of the spatial function of the dynamic displacement of cables. However, if only the effect of bending stiffness is considered, Eq. (13) will have the following aspect:

$$w(x) = C_1 \cosh(x\kappa) + C_2 \sinh(x\kappa) + C_3 \cos(x\kappa) + C_4 \sin(x\kappa) \tag{14}$$

in which $\kappa = \delta = \beta$. Eq. (14) is presented by Clough and Penzien (1975).

When considering the harmonic solution "$\sin(\omega t + \phi)$" to the function of time "$q(t)$" it is possible to obtain the equation representing the dynamic displacement of a beam as a function of the constants C_1, C_2, C_3, e C_4 that must be determined from the boundary conditions. Thus, one has that:

$$v(x,t) = \sin(\omega t + \phi)[C_1 \cosh(x\delta) + C_2 \sinh(x\delta) + C_3 \cos(x\beta) + C_4 \sin(x\beta)] \tag{15}$$

The next section presents, from Bernoulli's beam theory, the analytical solutions of the dynamic equation of the motion of beams to four different types of boundary conditions: pinned-pinned, fixed-pinned, fixed-free and fixed-fixed.

3 DYNAMIC EQUATION OF MOTION OF BEAMS ACCORDING BERNOULLI'S BEAM THEORY

Bernoulli's beam theory is based on the assumption that the beam's cross sections remain flat and perpendicular to the reference axis, after deformation occurs. Therefore, it is possible to write:

$$\tan[\theta(x,t)] \cong \theta(x,t) = \frac{\partial v(x,t)}{\partial x} \tag{16}$$

in which $\theta(x,t)$ represents the rotation of a generic cross section and $v(x,t)$ is the vertical displacement.

Equation (16) is substituted in equations (1) and (4). Furthermore, combining these results with the first derivative of Eq. (3) with respect to *x*, and with Eq. (2), using the assumption represented by Eq. (16), results in:

$$EI\frac{\mathrm{d}^4 v(x,t)}{\mathrm{d}x^4}+m\frac{\partial^2 v(x,t)}{\partial t^2}-\frac{mI}{A}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 v(x,t)}{\partial t^2}\right)=0 \tag{17}$$

Using the MSV with $q(t)=\sin(\omega t+\phi)$ the following expression can be obtained:

$$\sin(\omega t+\phi)\left[w^{\text{iv}}(x)-\frac{m\omega^2}{EI}w(x)+\frac{m\omega^2}{EA}w''(x)\right]=0 \tag{18}$$

in which "'" indicates the order of derivatives with respect to independent variable *x*, while "."indicates the order of derivatives with respect to independent variable *t*.

Avoiding the trivial solution, i.e., "$\sin(\omega t+\phi)=0$" and using the differential operator (D = d/dt), Eq. (18) can be rewritten as:

$$\left[D^4+0D^3+\left(\frac{m\omega^2}{EA}\right)D^2+0D+\left(-\frac{m\omega^2}{EI}\right)\right]w(x)=0 \tag{19}$$

Comparing Eq. (19) with Eq. (6), one can conclude that:

$$B=\frac{m\omega^2}{EA}\ \text{and}\ R=\frac{m\omega^2}{EI} \tag{20}$$

With the values of B and R known it is possible to determine the values of δ and β shown in Eq. (11) and consequently, the dynamic displacement of the beams, as shown in Eq. (15). The next four items show, from Bernoulli's beam theory, analytical expressions of the dynamic displacement of beams for the four boundary conditions studied in this paper.

3.1 Pinned-pinned beam

Considering that x and y are Cartesian Coordinates, and $v(x,t)$ is the dynamic displacement. The boundary conditions for this case considering Bernoulli's theory are:

- $v(0,t)=0 \therefore \sin(\omega t+\phi)w(0)=0 \therefore w(0)=0,$ at $x=0$;
- $M(0,t)=0 \therefore v''(0,t) \therefore \sin(\omega t+\phi)w''(0)=0 \therefore w''(0)=0,$ at $x=0$;
- $v(L,t)=0 \therefore \sin(\omega t+\phi)w(L)=0 \therefore w(L)=0,$ at $x=L$;
- $M(L,t)=0 \therefore v''(L,t)=0 \therefore \sin(\omega t+\phi)w''(L)=0 \therefore w''(L)=0,$ at $x=L$.

It is important to remember that with the boundary conditions shown above, the trivial solution "$\sin(\omega t+\phi)$" is always avoided.

For $w(0)=0$ it follows that:

$$C_1=-C_3 \tag{21}$$

For $w''(0)=0$ it follows that:

$$C_3\left(\beta^2+\delta^2\right)=0 \tag{22}$$

As it is not necessary to consider $\beta^2=-\delta^2$, one has:

$$C_1=C_3=0 \tag{23}$$

For $w(L)=0$ it follows that:

$$C_2 = -C_4 \frac{\sin(L\beta)}{\sinh(L\delta)} = 0 \tag{24}$$

For $w''(L) = 0$ it follows that:

$$C_2 \sinh(L\delta)\delta^2 - C_4 \sin(L\beta)\beta^2 = 0 \tag{25}$$

Substituting Eq. (22) into Eq. (23):

$$-C_4 \sin(L\beta)\delta^2 - C_4 \sin(L\beta)\beta^2 = 0 \tag{26}$$

Avoiding the trivial solution, i.e., "$C_4 = 0$", one has:

$$\sin(L\beta)\left(\delta^2 + \beta^2\right) = 0 \tag{27}$$

Eq. (25) is the frequency function for the case of pinned-pinned boundary conditions for Bernoulli's beam theory. Substitution of Eq. (23), and Eq. (24) into Eq. (13), leads to:

$$w(x) = C_4\left(\sin(x\beta) - \sinh(x\delta)\frac{\sin(L\beta)}{\sinh(L\delta)}\right) \tag{28}$$

Finally, the solution for the displacements of the pinned-pinned beam according to Bernoulli's theory and considering the contribution of *n* modes is given by:

$$v(x,t) = C_4 \sum_{i=1}^{n} \sin(\omega_i\, t + \phi_i)\left(\sin(x\beta_i) - \sinh(x\delta_i)\frac{\sin(L\beta_i)}{\sinh(L\delta_i)}\right) \tag{29}$$

The next boundary condition case that will be presented is the fixed-free beam.

3.2 Fixed-free beam

For this case considering Bernoulli's theory are:

- $v(0,t) = 0 \therefore \sin(\omega t + \phi)w(0) = 0 \therefore w(0) = 0,$ at $x = 0$;
- $v'(0,t) = 0 \therefore \sin(\omega t + \phi)w'(0) = 0 \therefore w'(0) = 0,$ at $x = 0$;
- $M(L,t) = 0 \therefore v''(L,t) = 0 \therefore \sin(\omega t + \phi)w''(L) = 0 \therefore w''(L) = 0,$ at $x = L$;
- $Q(L,t) = 0,$ at $x = L$.

It is important to remember that for the boundary conditions shown above, the trivial solution "$\sin(\omega t + \phi)$" is always avoided. For the boundary condition $Q(L,t) = 0$ it is important to observe that, due to the fact that Bernoulli's beam theory does not predict the effect of angular distortion "γ_{xy}", Eq. (5) cannot be used to relate the shear with vertical displacement "$v(L,t)$". However, it is possible to rewrite Eq. (2) as:

$$Q(L) = -m\omega^2 \int w(L)\,\mathrm{d}x \tag{30}$$

in which:

$$\int w(L)\,\mathrm{d}x = C_1 \frac{\sinh(x\delta)}{\delta} + C_2 \frac{\cosh(x\delta)}{\delta} + C_3 \frac{\sin(x\beta)}{\beta} - C_4 \frac{\cos(x\beta)}{\beta} \tag{31}$$

Equation (30) will be used for the last boundary condition of the fixed-free beam. From the first boundary condition, $w(0) = 0$, it follows that:

$$C_1 = -C_3 \tag{32}$$

Equation (32) is equal to Eq. (21), as expected. From the second boundary condition, $w'(0) = 0$, it follows that:

$$C_2 = -C_4 \frac{\beta}{\delta} \tag{33}$$

From the third boundary condition, $w''(L) = 0$, it follows that:

$$-C_3\left[\cos(L\beta)\beta^2 + \cosh(L\delta)\delta^2\right] - C_4\left[\sin(L\beta)\beta^2 + \sinh(L\delta)\delta\beta\right] = 0 \tag{34}$$

Equation (34) can be rewritten as:

$$C_3 = -C_4 \alpha 1 \tag{35}$$

in which:

$$\alpha 1 = \frac{\left[\sin(L\beta)\beta^2 + \sinh(L\delta)\delta\beta\right]}{\left[\cos(L\beta)\beta^2 + \cosh(L\delta)\delta^2\right]} \tag{36}$$

From the fourth boundary condition, Eq. (30), it follows that:

$$-C_4\left\{\alpha 1[\delta\sin(L\beta) - \beta\sinh(L\delta)] + \delta\cos(L\beta) + \frac{\beta^2}{\delta}\cosh(L\delta)\right\} = 0 \tag{37}$$

Avoiding the trivial solution, i.e., "$C_4 = 0$", one has:

$$\alpha 1[\delta\sin(L\beta) - \beta\sinh(L\delta)] + \delta\cos(L\beta) + \frac{\beta^2}{\delta}\cosh(L\delta) = 0 \tag{38}$$

Equation (38) is the frequency function of the fixed-free beam for Bernoulli's beam theory. Substituting Eq. (32), Eq. (33), and Eq. (35) into Eq. (13), results:

$$w(x) = C_4\left\{\alpha 1[\cosh(x\delta) - \cos(x\beta)] - \frac{\beta}{\delta}\sinh(x\delta) + \sin(x\beta)\right\} \tag{39}$$

Finally, the solution for the displacements of the fixed-free beam according to Bernoulli's theory and considering the contribution of n modes is given by:

$$v(x,t) = C_4 \sum_{i=1}^{n} \sin(\omega_i\, t + \phi_i)\left(\alpha 1_i[\cosh(x\delta_i) - \cos(x\beta_i)] - \frac{\beta_i}{\delta_i}\sinh(x\delta_i) + \sin(x\beta_i)\right) \tag{40}$$

The next boundary condition case that will be presented is the fixed-pinned beam.

3.3 Fixed-pined beam

For this case considering Bernoulli's theory are:

- $v(0,t) = 0 \therefore \sin(\omega t + \phi)w(0) = 0 \therefore w(0) = 0,$ at $x = 0$;
- $v'(0,t) = 0 \therefore \sin(\omega t + \phi)w'(0) = 0 \therefore w'(0) = 0,$ at $x = 0$;
- $v(L,t) = 0 \therefore \sin(\omega t + \phi)w(L) = 0 \therefore w(L) = 0,$ at $x = L$;
- $M(L,t) = 0 \therefore v''(L,t) = 0 \therefore \sin(\omega t + \phi)w''(L) = 0 \therefore w''(L) = 0,$ at $x = L$.

It is important to remember that for the boundary conditions shown above, the trivial solution "$\sin(\omega t+\phi)$" is always avoided.

From the first boundary condition, $w(0)=0$, it follows that:

$$C_1=-C_3 \tag{41}$$

Equation (41) is equal to Eq. (21) and Eq. (32), as expected. From the second boundary condition, $w'(0)=0$, it follows that:

$$C_2=-C_4\frac{\beta}{\delta} \tag{42}$$

Equation (42) is equal to Eq. (33), as expected. From the third boundary condition, $w(L)=0$, it follows that:

$$C_3[\cos(L\beta)-\cosh(L\delta)]+C_4\left[\sin(L\beta)-\sinh(L\delta)\frac{\beta}{\delta}\right]=0 \tag{43}$$

Equation (43) can be rewritten as:

$$C_3=-C_4\alpha 1 \tag{44}$$

in which:

$$\alpha 1=\frac{\sin(L\beta)-\sinh(L\delta)\dfrac{\beta}{\delta}}{\cos(L\beta)-\cosh(L\delta)} \tag{45}$$

From the fourth boundary condition, $w''(L)=0$, it follows that:

$$-C_4\left\{\alpha 1\left[\cos(L\beta)\beta^2+\cosh(L\delta)\delta^2\right]-\beta[\beta\sin(L\beta)+\delta\sinh(L\delta)]\right\}=0 \tag{46}$$

Avoiding the trivial solution, i.e., "$C_4=0$", one has:

$$\alpha 1\left[\cos(L\beta)\beta^2+\cosh(L\delta)\delta^2\right]-\beta[\beta\sin(L\beta)+\delta\sinh(L\delta)]=0 \tag{47}$$

Equation (47) is the frequency function of the pinned-pinned beam. for Bernoulli's beam theory. Substituting Eq. (41), Eq. (42), and Eq. (44) into Eq. (13), results:

$$w(x)=C_4\left\{\alpha 1[\cosh(x\delta)-\cos(x\beta)]-\frac{\beta}{\delta}\sinh(x\delta)+\sin(x\beta)\right\} \tag{48}$$

Finally, the solution for the displacements of the fixed-pinned beam according to Bernoulli's theory and considering the contribution of n modes is given by:

$$v(x,t)=C_4\sum_{i=1}^{n}\sin(\omega_i t+\phi_i)\left\{\alpha 1_i[\cosh(x\delta_i)-\cos(x\beta_i)]-\frac{\beta_i}{\delta_i}\sinh(x\delta_i)+\sin(x\beta_i)\right\} \tag{49}$$

The next boundary condition case that will be presented is the fixed-fixed beam.

3.4 Fixed-fixed beam

For this case considering Bernoulli's theory are:

- $v(0,t)=0 \therefore \sin(\omega t+\phi)w(0)=0 \therefore w(0)=0,\ \text{at } x=0$;
- $v'(0,t)=0 \therefore \sin(\omega t+\phi)w'(0)=0 \therefore w'(0)=0,\ \text{at } x=0$;
- $v(L,t)=0 \therefore \sin(\omega t+\phi)w(L)=0 \therefore w(L)=0,\ \text{at } x=L$;
- $v'(L,t)=0 \therefore \sin(\omega t+\phi)w'(L)=0 \therefore w'(L)=0,\ \text{at } x=L$;

From the first boundary condition, $w(0)=0$, it follows that:

$$C_1=-C_3 \tag{50}$$

Equation (50) is equal to Eq. (21), Eq. (32) and Eq. (41), as expected. From the second boundary condition, $w'(0)=0$, it follows that:

$$C_2=-C_4\frac{\beta}{\delta} \tag{51}$$

Equation (51) is equal to Eq. (33) and Eq. (42), as expected. From the third boundary condition, $w(L)=0$, it follows that:

$$C_3[\cos(L\beta)-\cosh(L\delta)]+C_4\left[\sin(L\beta)-\sinh(L\delta)\frac{\beta}{\delta}\right]=0 \tag{52}$$

Equation (52) is equal to Eq. (44), as expected. So, the relation between constants C_3 and C_4 for the fixed-fixed beam is equal to the fixed-pinned beam, which is given by Eq. (44).

From the fourth boundary condition, $w'(L)=0$, it follows that:

$$C_4\{\alpha 1[\sinh(L\delta)\delta+\sin(L\beta)\beta]-\beta\cosh(L\delta)+\beta\cos(L\beta)\}=0 \tag{53}$$

Avoiding the trivial solution, i.e., "$C_4=0$", one has:

$$\alpha 1[\sinh(L\delta)\delta+\sin(L\beta)\beta]-\beta\cosh(L\delta)+\beta\cos(L\beta)=0 \tag{54}$$

Equation (54) is the frequency function of the fixed-fixed beam. for Bernoulli's beam theory. Substituting Eq. (50), Eq. (51), and Eq. (52) into Eq. (13), results:

$$w(x)=C_4\left\{\alpha 1[\cosh(x\delta)-\cos(x\beta)]-\frac{\beta}{\delta}\sinh(x\delta)+\sin(x\beta)\right\} \tag{55}$$

Finally, the solution for the displacements of the fixed-fixed beam according to Bernoulli's theory and considering the contribution of *n* modes is given by:

$$v(x,t)=C_4\sum_{i=1}^{n}\sin(\omega_i t+\phi_i)\left\{\alpha 1_i[\cosh(x\delta_i)-\cos(x\beta_i)]-\frac{\beta_i}{\delta_i}\sinh(x\delta_i)+\sin(x\beta_i)\right\} \tag{56}$$

Next section of this paper shows the obtaining of the displacement and of the rotation dynamics according to Timoshenko's beam theory.

4 DYNAMIC EQUATION OF MOTION OF BEAMS ACCORDING TO TIMOSHENKO'S BEAM THEORY

Timoshenko's beam theory is based on the assumption that:

$$\gamma_{xy}(x,t) = \frac{\partial v(x,t)}{\partial x} - \theta(x,t) \tag{57}$$

Substituting Eq. (57) and Eq. (5) in Eq. (2) results in:

$$\theta'(x,t) = v''(x,t) - \frac{m}{GA\kappa}\ddot{v}(x,t) \tag{58}$$

Integrating in both sides, one has:

$$\theta(x,t) = v'(x,t) - \frac{m}{GA\kappa}\int \ddot{v}(x,t)\,\mathrm{d}x \tag{59}$$

The expressions of $v'(x,t)$ and $\int \ddot{v}(x,t)\,\mathrm{d}x$ can be obtained from Eq. (13). Therefore, using the MSV, and assuming a harmonic solution for the function $q(t)$, the Eq. (60) can be rewritten as:

$$\theta(x,t) = C_4 \sum_{i=1}^{n} \left\{ \begin{array}{l} \sin(\omega_i\, t + \phi_i)\big[\big(C_1\ \delta_i \sinh(x\delta_i) + C_2\ \delta_i \cosh(x\delta_i) - \\ C_3\ \beta_i \sin(x\beta_i) + C_4\ \beta_i \cos(x\beta_i)\big) + \\ \dfrac{m\omega^2}{GA\kappa}\left(\dfrac{C_1}{\delta_i}\sinh(x\delta_i) + \dfrac{C_2}{\delta_i}\cosh(x\delta_i) + \right. \\ \left. \dfrac{C_3}{\beta_i}\sin(x\beta_i) - \dfrac{C_4}{\beta_i}\cos(x\beta_i)\right)\Big] \end{array} \right\} \tag{60}$$

Equation (60) is the dynamic rotation of beams. To obtain the differential equation in terms of $v(x,t)$ for the case in which one uses Timoshenko's beam theory, it is necessary to replace Eq. (2) and Eq. (4) in the first derivative of Eq. (3) with respect to *x*. In response it follows that:

$$EI\frac{\partial^3\theta(x,t)}{\partial x^3} + m\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{mI}{A}\frac{\partial^2\theta'(x,t)}{\partial t^2} = 0 \therefore EI\theta'''(x,t) + m\ddot{v}(x,t) - \frac{mI}{A}\frac{\partial^2\theta'(x,t)}{\partial t^2} = 0 \tag{61}$$

Deriving Eq. (58) with respect to *x* twice, it follows that:

$$\theta'''(x,t) = v^{\mathrm{iv}}(x,t) - \frac{m}{GA\kappa}\frac{\partial^2\ddot{v}(x,t)}{\partial x^2} \tag{62}$$

Substituting Eq. (58) and Eq. (62) in Eq. (61), it follows that:

$$EIv^{\mathrm{iv}}(x,t) - \frac{mEI}{GA\kappa}\frac{\partial^2\ddot{v}(x,t)}{\partial x^2} + m\ddot{v}(x,t) - \frac{mI}{A}\frac{\partial^2\ddot{v}(x,t)}{\partial x^2} + \frac{m^2 I}{GA^2\kappa}\frac{\partial^4 v(x,t)}{\partial t^4} = 0 \tag{63}$$

in which $v(x,t)$ is unknown. Dividing Eq. (63) by *EI* and using the MSV with a harmonic solution, it follows that:

$$\sin(\omega t + \phi)\left[w^{\mathrm{iv}}(x) + \frac{m\omega^2}{GA\kappa}w''(x) - \frac{m\omega^2}{EI}w(x) + \frac{m\omega^2}{EA}w''(x) + \frac{m^2\omega^4}{EGA^2\kappa}w(x)\right] = 0 \tag{64}$$

Avoiding the trivial solution, and using the differential operator (D = d/dt), Eq. (64) can be rewritten as:

$$\left[D^4 + 0D^3 + \left(\frac{m\omega^2}{GA\kappa} + \frac{m\omega^2}{EA}\right)D^2 + 0D + \left(\frac{m^2\omega^4}{EGA^2\kappa} - \frac{m\omega^2}{EI}\right)\right]w(x) = 0 \tag{65}$$

Comparing Eq. (65) with Eq. (6), one can conclude that:

$$B = \frac{m\omega^2}{GA\kappa} + \frac{m\omega^2}{EA} \text{ and } R = \frac{m^2\omega^4}{EGA^2\kappa} - \frac{m\omega^2}{EI} \tag{66}$$

With the values of *B* and *R* known it is possible to determine the values of δ and β shown in Eq. (11) and consequently, the dynamic displacement of beams, as shown in Eq. (15). The next four items show, from Timoshenko's beam theory, analytical expressions of the dynamic displacement of beams for the four boundary conditions studied in this paper.

4.1 Pinned-pinned beam

The boundary conditions for this case considering Timoshenko's theory are:

- $v(0,t) = 0 \therefore \sin(\omega t + \phi)w(0) = 0 \therefore w(0) = 0,$ at $x = 0$;
- $M(0,t) = 0 \therefore EI\theta'(0,t) = 0 \therefore \theta'(0,t) = 0,$ at $x = 0$;
- $v(L,t) = 0 \therefore \sin(\omega t + \phi)w(L) = 0 \therefore w(L) = 0,$ at $x = L$;
- $M(L,t) = 0 \therefore EI\theta'(L,t) = 0 \therefore \theta'(L,t) = 0,$ at $x = L$.

From the first boundary condition, $w(0) = 0$, it follows that:

$$C_1 = -C_3 \tag{67}$$

Equation (67) is equal to all other equations corresponding to the boundary condition $w(0) = 0$, regardless of the beam's theory used. From the second boundary condition, $\theta'(0,t) = 0,$ it follows that:

$$C_1\left(G\,A\,\kappa\,\delta^2 + m\,\omega^2\right) - C_3\left(G\,A\,\kappa\,\beta^2 - m\,\omega^2\right) = 0 \therefore -C_3\,G\,A\,\kappa\left(\beta^2 + \delta^2\right) = 0 \tag{68}$$

Assuming that "$\left(\beta^2 + \delta^2\right)$" is not necessarily equal to 0, one has that:

$$C_1 = C_3 = 0 \tag{69}$$

From the third boundary condition, $w(L) = 0$, it follows that:

$$C_2 = -C_4\,\alpha 1 \tag{70}$$

in which:

$$\alpha 1 = \frac{\sin(L\beta)}{\sinh(L\delta)} \tag{71}$$

From the fourth boundary condition, $\theta'(L,t) = 0,$ it follows that:

$$C_4\left\{\sin(L\beta)\left[\beta^2 + \delta^2\right]\right\} \tag{72}$$

Avoiding the trivial solution, i.e., "$C_4 = 0$", one has:

$$\sin(L\beta)\left[\delta^2 + \beta^2\right] = 0 \tag{73}$$

Equation (73) is the frequency function of the pinned-pinned beam for Timoshenko's beam theory. Substituting Eq. (69) and Eq. (70) into Eq. (13), results:

$$w(x) = C_4\{\sin(x\beta) - \sin(L\beta)\} \tag{74}$$

Finally, the solution for the displacements of the pinned-pinned beam according to Timoshenko's theory and considering the contribution of n modes is given by:

$$v(x,t) = C_4 \sum_{i=1}^{n} \sin(\omega_i\, t + \phi_i)\{\sin(x\beta_i) - \sin(L\beta_i)\} \tag{75}$$

The solution for the rotations of the pinned-pinned beam according to Timoshenko's theory and considering the contribution of n modes is given by:

$$\theta(x,t) = C_4 \sum_{i=1}^{n} \sin(\omega_i\, t + \phi_i) \left\{ \begin{array}{l} \beta_i \cos(x\beta_i) - \dfrac{\sin(L\beta_i)}{\sinh(L\delta_i)} \delta_i \cosh(x\delta_i) - \\ \dfrac{m\,\omega_i^2}{G\,A\,\kappa} \left[\dfrac{\cos(x\beta_i)}{\beta_i} + \dfrac{\sin(L\beta_i)}{\sinh(L\delta_i)} \dfrac{\cosh(x\delta_i)}{\delta_i} \right] \end{array} \right\} \tag{76}$$

The next boundary condition case that will be presented is the fixed-free beam.

4.2 Fixed-free beam

The boundary conditions for this case considering Timoshenko's theory are:

- $v(0,t) = 0 \therefore \sin(\omega t + \phi) w(0) = 0 \therefore w(0) = 0,$ at $x = 0$;
- $\theta(0,t) = 0,$ at $x = 0$;
- $\theta'(L,t) = 0,$ at $x = L$;
- $\gamma_{xy}(L,t) = 0 \therefore v'(L,t) - \theta(L,t) = 0,$ at $x = L$.

From the first boundary condition, $w(0) = 0$, it follows that:

$$C_1 = -C_3 \tag{77}$$

From the second boundary condition, $\theta(0,t) = 0,$ it follows that:

$$C_2 = C_4\,\alpha 1 \tag{78}$$

in which $\alpha 1$ is given by:

$$\alpha 1 = \frac{\delta\, m\, \omega^2 - G\,A\,\kappa\,\delta\,\beta^2}{\beta\, m\, \omega^2 + G\,A\,\kappa\,\beta\,\delta^2} \tag{79}$$

From the third boundary condition, $\theta'(L,t) = 0,$ it follows that:

$$C_3 = -C_4\,\alpha 2 \tag{80}$$

in which:

$$\alpha 2 = \frac{\alpha 1 \sinh(L\delta)\left[m\,\omega^2 + G\,A\,\kappa\,\delta^2\right] + \sin(L\beta)\left[m\,\omega^2 - G\,A\,\kappa\,\beta^2\right]}{\cos(L\beta)\left[m\,\omega^2 - G\,A\,\kappa\,\beta^2\right] - \cosh(L\delta)\left[m\,\omega^2 + G\,A\,\kappa\,\delta^2\right]} \tag{81}$$

From the fourth boundary condition, $v'(L,t) - \theta(L,t) = 0,$ it follows that:

$$C_4\{\alpha 1\,\beta\cosh(L\delta)+\alpha 2[\beta\sinh(L\delta)-\delta\sin(L\beta)]-\delta\cos(L\beta)\} \tag{82}$$

Avoiding the trivial solution, i.e., "$C_4 = 0$", one has:

$$\alpha 1\,\beta\cosh(L\delta)+\alpha 2[\beta\sinh(L\delta)-\delta\sin(L\beta)]-\delta\cos(L\beta) \tag{83}$$

Equation (83) is the frequency function of the fixed-free beam for Timoshenko's beam theory. Substituting Eq. (77), Eq. (78), and Eq. (80) into Eq. (13), results:

$$w(x)=C_4\{\alpha 1\sinh(x\delta)+\alpha 2[\cosh(x\delta)-\cos(x\beta)]+\sin(x\beta)\} \tag{84}$$

Finally, the solution for the displacements of the fixed-free beam according to Timoshenko's theory and considering the contribution of *n* modes is given by:

$$v(x,t)=C_4\sum_{i=1}^{n}\sin(\omega_i\,t+\phi_i)\{\alpha 1_i\sinh(x\delta_i)+\alpha 2_i[\cosh(x\delta_i)-\cos(x\beta_i)]+\sin(x\beta_i)\} \tag{85}$$

The solution for the rotations of the fixed-free beam according to Timoshenko's theory and considering the contribution of *n* modes is given by:

$$\theta(x,t)=C_4\sum_{i=1}^{n}\sin(\omega_i\,t+\phi_i)\left\{\begin{array}{l}\alpha 1_i\,\delta_i\cosh(x\delta_i)+\alpha 2_i[\delta_i\sinh(x\delta_i)+\beta_i\sin(x\beta_i)]+\\ \beta_i\cos(x\beta_i)+\dfrac{m\,\omega_i^2}{G\,A\,\kappa}\left[\alpha 1_i\dfrac{\cosh(x\delta_i)}{\delta_i}+\right.\\ \left.\alpha 2_i\left[\dfrac{\sinh(x\delta_i)}{\delta_i}-\dfrac{\sin(x\beta_i)}{\beta_i}\right]-\dfrac{\cos(x\beta_i)}{\beta_i}\right]\end{array}\right\} \tag{86}$$

The next boundary condition case that will be presented is the fixed-pinned beam.

4.3 Fixed-pinned beam

The boundary conditions for this case considering Timoshenko's theory are:

- $v(0,t)=0 \therefore \sin(\omega t+\phi)w(0)=0 \therefore w(0)=0$, at $x=0$;
- $\theta(0,t)=0$, at $x=0$;
- $v(L,t)=0 \therefore \sin(\omega t+\phi)w(L)=0 \therefore w(L)=0$, at $x=L$;
- $\theta'(L,t)=0$, at $x=L$ at $x=L$.

From the first boundary condition, $w(0)=0$, it follows that:

$$C_1=-C_3 \tag{87}$$

From the second boundary condition, $\theta(0,t)=0$, it follows that:

$$C_2=C_4\,\alpha 1 \tag{88}$$

in which $\alpha 1$ is given by Eq. (79).

From the third boundary condition, $w(L)=0$, it follows that:

$$C_3=-C_4\,\alpha 2 \tag{89}$$

in which:

$$\alpha 2=\frac{\alpha 1\sinh(L\delta)+\sin(L\beta)}{\cos(L\beta)-\cosh(L\delta)} \tag{90}$$

From the fourth boundary condition, $\theta'(L,t)=0,$ it follows that:

$$C_4\left\{\alpha 1\sinh(L\delta)\left[m\,\omega^2+G\,A\,\kappa\,\delta^2\right]+\sin(L\beta)\left[m\,\omega^2-G\,A\,\kappa\,\beta^2\right]-\alpha 2\left[\cos(L\beta)\left(m\,\omega^2-G\,A\,\kappa\,\beta^2\right)-\cosh(L\delta)\left(m\,\omega^2+G\,A\,\kappa\,\delta^2\right)\right]\right\} \tag{91}$$

Avoiding the trivial solution, i.e., "$C_4=0$", one has:

$$\alpha 1\sinh(L\delta)\left[m\,\omega^2+G\,A\,\kappa\,\delta^2\right]+\sin(L\beta)\left[m\,\omega^2-G\,A\,\kappa\,\beta^2\right]-\alpha 2\left[\cos(L\beta)\left(m\,\omega^2-G\,A\,\kappa\,\beta^2\right)-\cosh(L\delta)\left(m\,\omega^2+G\,A\,\kappa\,\delta^2\right)\right] \tag{92}$$

Equation (92) is the frequency function of the fixed-pinned beam for Timoshenko's beam theory. Substituting Eq. (87), Eq. (88), and Eq. (89) into Eq. (13), results:

$$w(x)=C_4\{\alpha 1\sinh(x\delta)+\alpha 2[\cosh(x\delta)-\cos(x\beta)]+\sin(x\beta)\} \tag{93}$$

Finally, the solution for the displacements of the fixed-pinned beam according to Timoshenko's theory and considering the contribution of n modes is given by:

$$v(x,t)=C_4\sum_{i=1}^{n}\sin(\omega_i\,t+\phi_i)\{\alpha 1_i\sinh(x\delta_i)+\alpha 2_i[\cosh(x\delta_i)-\cos(x\beta_i)]+\sin(x\beta_i)\} \tag{94}$$

The solution for the rotations of the fixed-pinned beam according to Timoshenko's theory and considering the contribution of n modes is given by:

$$\theta(x,t)=C_4\sum_{i=1}^{n}\sin(\omega_i\,t+\phi_i)\left\{\begin{aligned}&\alpha 1_i\,\delta_i\cosh(x\delta_i)+\alpha 2_i[\delta_i\sinh(x\delta_i)+\beta_i\sin(x\beta_i)]+\\&\beta_i\cos(x\beta_i)+\frac{m\,\omega_i^2}{G\,A\,\kappa}\left[\alpha 1_i\frac{\cosh(x\delta_i)}{\delta_i}+\right.\\&\left.\alpha 2_i\left[\frac{\sinh(x\delta_i)}{\delta_i}-\frac{\sin(x\beta_i)}{\beta_i}\right]-\frac{\cos(x\beta_i)}{\beta_i}\right]\end{aligned}\right\} \tag{95}$$

The next boundary condition case that will be presented is the fixed-fixed beam.

4.4 Fixed-fixed beam

The boundary conditions for this case considering Timoshenko's theory are:

- $v(0,t)=0\;\therefore\;\sin(\omega t+\phi)w(0)=0\;\therefore\;w(0)=0,$ at $x=0$;
- $\theta(0,t)=0,$ at $x=0$;
- $v(L,t)=0\;\therefore\;\sin(\omega t+\phi)w(L)=0\;\therefore\;w(L)=0,$ at $x=L$;
- $\theta(L,t)=0,$ at $x=L$ at $x=L$.

From the first boundary condition, $w(0)=0$, it follows that:

$$C_1 = -C_3 \tag{96}$$

From the second boundary condition, $\theta(0,t) = 0,$ it follows that:

$$C_2 = C_4\, \alpha 1 \tag{97}$$

in which $\alpha 1$ is given by Eq. (79).

From the third boundary condition, $w(L) = 0,$ it follows that:

$$C_3 = -C_4\, \alpha 2 \tag{98}$$

in which $\alpha 2$ is given by Eq. (90). From the fourth boundary condition, $\theta(L,t) = 0,$ it follows that:

$$\begin{aligned} C_4 \Bigg\{ & \alpha 1 \cosh(L\delta)\left[\delta + \frac{m\,\omega^2}{G\,A\,\kappa\,\delta}\right] + \cos(L\beta)\left[\beta - \frac{m\,\omega^2}{G\,A\,\kappa\,\beta}\right] - \\ & \alpha 2\left[\sin(L\beta)\left(\frac{m\,\omega^2}{G\,A\,\kappa\,\beta} - \beta\right) - \sinh(L\delta)\left(\frac{m\,\omega^2}{G\,A\,\kappa\,\delta} + \delta\right)\right]\Bigg\} \end{aligned} \tag{99}$$

Avoiding the trivial solution, i.e., "$C_4 = 0$", one has:

$$\begin{aligned} & \alpha 1 \cosh(L\delta)\left[\delta + \frac{m\,\omega^2}{G\,A\,\kappa\,\delta}\right] + \cos(L\beta)\left[\beta - \frac{m\,\omega^2}{G\,A\,\kappa\,\beta}\right] - \\ & \alpha 2\left[\sin(L\beta)\left(\frac{m\,\omega^2}{G\,A\,\kappa\,\beta} - \beta\right) - \sinh(L\delta)\left(\frac{m\,\omega^2}{G\,A\,\kappa\,\delta} + \delta\right)\right] \end{aligned} \tag{100}$$

Equation (100) is the frequency function of the fixed-fixed beam for Timoshenko's beam theory. Substituting Eq. (95), Eq. (96), and Eq. (97) into Eq. (13), results:

$$w(x) = C_4\{\alpha 1 \sinh(x\delta) + \alpha 2[\cosh(x\delta) - \cos(x\beta)] + \sin(x\beta)\} \tag{101}$$

Finally, the solution for the displacements of the fixed-fixed beam according to Timoshenko's theory and considering the contribution of *n* modes is given by:

$$v(x,t) = C_4 \sum_{i=1}^{n} \sin(\omega_i\, t + \phi_i)\{\alpha 1_i \sinh(x\delta_i) + \alpha 2_i[\cosh(x\delta_i) - \cos(x\beta_i)] + \sin(x\beta_i)\} \tag{102}$$

The solution for the rotations of the fixed-fixed beam according to Timoshenko's theory and considering the contribution of *n* modes is given by:

$$\theta(x,t) = C_4 \sum_{i=1}^{n} \sin(\omega_i\, t + \phi_i) \left\{ \begin{array}{l} \alpha 1_i\, \delta_i \cosh(x\delta_i) + \alpha 2_i [\delta_i \sinh(x\delta_i) + \beta_i \sin(x\beta_i)] + \\ \beta_i \cos(x\beta_i) + \dfrac{m\, \omega_i^2}{G\, A\, \kappa} \left[\alpha 1_i \dfrac{\cosh(x\delta_i)}{\delta_i} + \right. \\ \left. \alpha 2_i \left[\dfrac{\sinh(x\delta_i)}{\delta_i} - \dfrac{\sin(x\beta_i)}{\beta_i} \right] - \dfrac{\cos(x\beta_i)}{\beta_i} \right] \end{array} \right\} \quad (103)$$

5 CONCLUSIONS

This paper is divided into two parts. From the hypothesis that beams are axially inextensible, Part 1 of this work presented the differential equation that governs the dynamic behavior of beams. Afterwards, the generic solution of beams was presented, which has four constants to be determined from the boundary conditions. Finally, the differential equations of the dynamic behavior of beams according to the Bernoulli and Timoshenko beam theories were shown.

Four boundary conditions were studied: pinned-pinned beam, fixed-free beam, fixed-pinned beam and fixed-fixed beam. In the case of Bernoulli's beam theory, the expressions of the vertical dynamic displacements for these four boundary conditions were presented. In the case of Timoshenko's beam theory, the expressions of the vertical dynamic displacements and the expressions of rotation dynamics for these four boundary conditions were presented.

Part 2 of this paper makes a comparison between the values of natural frequencies obtained by the analytical expressions presented in Part 1 and the values of natural frequencies obtained numerically.

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