CILAMCE
Ouro Preto/MG - Brazil
2003
XXIV IBERIAN LATIN-AMERICAN
CONGRESS ON COMPUTATIONAL
METHODS IN ENGINEERING

FORCE FORMULATION OF A NON-PRISMATIC TIMOSHENKO BEAM FINITE ELEMENT FOR DYNAMIC ANALYSIS OF FRAMES

Remo Magalhães de Souza
Associate Professor, Depto. de Construção Civil, Centro Tecnológico
Universidade Federal do Pará – Belém - Pa, CEP: 66.075-970, Brazil
remo@ufpa.br

Filip C. Filippou
Professor, Department of Civil and Environmental Engineering, 731 Davis Hall
University of California, Berkeley, CA 94720-1710, USA
filippou@ce.berkeley.edu

André Maués Brabo Pereira
Graduate Student in Civil Engineering – Escola Politécnica da Universidade de São Paulo
maues.pereira@poli.usp.br

Gandhy Y. R. Aranha Jr.
Graduate Student in Mechanical Engineering – Universidade Federal do Pará
gandhy@ufpa.br

Abstract. *This paper presents a force-based Timoshenko beam finite element for the linear dynamic analysis of frames containing members with variable cross-sections. The proposed formulation uses exact force interpolation functions, as opposed to the tradidionally adopted displacement interpolation functions, for the computation of the element stiffness matrix. This is accomplished through the computation of the element flexibility matrix in a system without rigid body modes. In addition, an exact expression is also determined for the displacement interpolation functions (shape functions), which allows for the computation of a consistent mass matrix for beams with variable cross-section, applying a simple numerical integration procedure. The computer implementation is discussed and some numerical examples are presented in order to demonstrate the method accuraccy and eficiency. The results obtained with the proposed formulation are compared with those obtained by other force and displacement formulations based on Timoshenko beam theory.*

Keywords: *Force Formulation, Beam Finite Element, Non-Prismatic beams, Consistent Mass Matrix, Timoshenko Beam Theory, Exact Shape Functions*

1. INTRODUCTION

The dynamic analysis of frames has been the subject of very extensive research over the last decades (Clough and Penzien, 1993; Chopra, 1995). However, the literature dealing specifically with dynamic analysis of non-prismatic beams is more limited, despite the large use of variable-cross-section systems in ancient and modern engineering structures.

Some recent studies on the dynamic analysis of non-prismatic members were performed by Chen (2000), Gajewski (2001), Chen (2002), and Ruta (2002). Chen (2000) studied vibration analyses of non-prismatic shear deformable beams resting on elastic foundations. Gajewski (2001) studied the problem of vibrations and stability of a non-prismatic column compressed by a follower force. Chen (2002) studied dynamic equilibrium equations of non-prismatic beams defined on an arbitrarily selected co-ordinate system. Ruta (2002) determined a dynamic stiffness matrix of a non-prismatic rod finite element resting on a two-parameter non-homogenous elastic foundation. It is important to emphasize that all of those studies were based on traditional displacement interpolation functions.

For the static case, the flexibility or force formulation is an excellent approach for the analysis of non-prismatic beams, since it renders exact element flexibility and stiffness matrices (Neuenhofer and Filippou 1997).

Due to the superiority of force-based elements for static nonlinear analysis of frames, several studies about the flexibility formulation have been carried out and presented in the literature (Backlund 1974, Carol and Murcia 1989, Spacone et al 1996, Neuenhofer and Filippou 1998, de Souza 1999).

The present paper is particularly concerned with the application of force formulations to the linear dynamic linear analysis of frames with non-prismatic beams.

A flexibility-based formulation of a mass matrix for the dynamic analysis of spatial frames consisting of curved elements with variable cross-sections was presented by Molins et al (1998). The main characteristic of the proposed formulation was the exact equilibrium of forces at any interior point of the beam, with no additional hypotheses about the distribution of displacements, strain or stresses.

The formulation proposed by Molins et al represents a very relevant achievement regarding the application of flexibility formulations in dynamic analysis, but it presents some shortcomings. The first problem is that the derived equation of the element mass matrix involves a triple integral (when an integration order n is chosen, this triple integral requires n^3 integration points located along the element axis). Another problem is that, due to numerical integration errors, the final results are not invariant with respect to node numbering (when the element connectivity is changed, i.e., the number of the end nodes are swapped, the analysis results change). However, as the order of integration increases, the problem related to element connectivity and node numbering tends to vanish, at the penalty of using a very large number of integration points.

The objective of the present study is the force formulation of a Timoshenko beam finite element for the linear dynamic analysis of planar frames with variable cross-sectional members, such that the problems discussed above can be overcome.

2. ELEMENT BASIC EQUATIONS

The proposed formulation is developed for straight planar beams with varying cross-section (as represented in Figure 1), with linear elastic material behavior (not necessarily homogeneous), and subjected to small deformations and displacements.

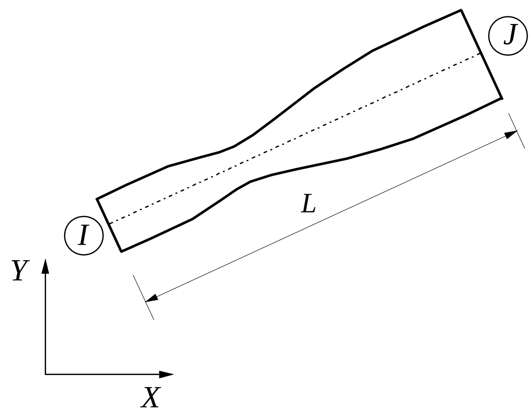

Figure 1 – Straight beam with varying cross-section.

2.1 Coordinate systems

The proposed element has two nodes, *I* and *J*, and can be represented by its reference axis, as shown in Figure 1. It is formulated considering three different reference systems, which are described next.

The global coordinate system (X, Y) is shown in Figure 2.a. The element has six degrees of freedom in this system, with nodal forces and nodal displacements grouped in vectors $\mathbf{p}^g$ and $\mathbf{d}^g$, respectively

$$\mathbf{p}^g = \left\langle p_1^g \quad p_2^g \quad p_3^g \quad p_4^g \quad p_5^g \quad p_6^g \right\rangle^{\mathrm{T}} \quad \mathbf{d}^g = \left\langle d_1^g \quad d_2^g \quad d_3^g \quad d_4^g \quad d_5^g \quad d_6^g \right\rangle^{\mathrm{T}} \tag{1}$$

As usual, the local coordinate system (x, y) is defined such that the x axis connects nodes *I* and *J*, according to Figure 2.b. The element also has six degrees of freedom in the local coordinate system, with nodal forces and displacements grouped in vectors $\mathbf{p}^l$ and $\mathbf{d}^l$, respectively

$$\mathbf{p}^l = \left\langle p_1^l \quad p_2^l \quad p_3^l \quad p_4^l \quad p_5^l \quad p_6^l \right\rangle^{\mathrm{T}} \quad \mathbf{d}^l = \left\langle d_1^l \quad d_2^l \quad d_3^l \quad d_4^l \quad d_5^l \quad d_6^l \right\rangle^{\mathrm{T}} \tag{2}$$

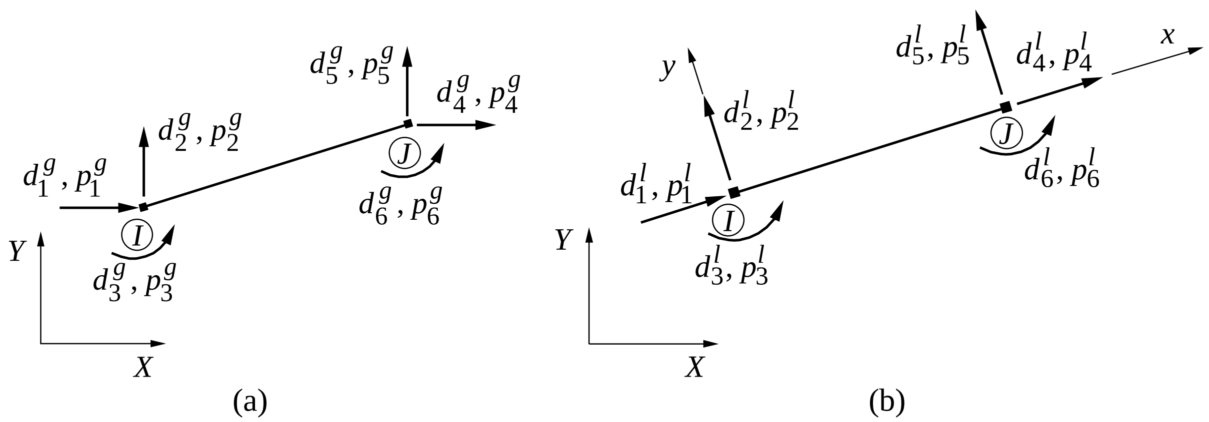

Figure 2 – Coordinate systems: (a) global; (b) local.

Due to the presence of rigid body modes in the local and global coordinate systems, the six components of each of the force vectors $\mathbf{p}^l$ and $\mathbf{p}^g$ are not independent, as they must satisfy equilibrium. Consequently, the stiffness matrix is singular in these systems, and has no inverse (i.e., there is no corresponding flexibility matrix). For this reason, the force-based element formulation is derived in another system, henceforth denoted basic coordinate system, which is free of rigid body modes. The number of degrees of freedom in the basic system is equal to the difference between the total number of degrees of freedom (six) and the number of rigid body modes (three).

There are many choices for such a reference system (a system free of rigid body modes). The proposed formulation adopts the system shown in Figure 3, which corresponds to a simply supported beam, defined by a straight line (chord) connecting the two displaced nodes *I* and *J*. It can be noted that the beam displacements measured with respect to the basic system (beam chord) do not include any rigid body modes.

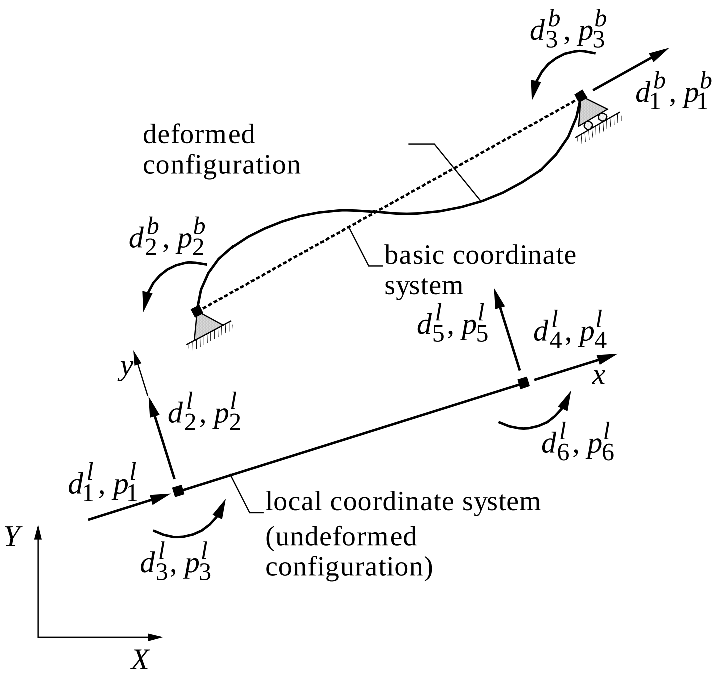

Figure 3 – Basic coordinate system

The basic system shown in Figure 3 has one axial displacement d_1^b and two rotations relative to the chord, d_2^b and d_3^b. These relative displacements correspond to the minimum number of geometric variables necessary to describe the deformation modes of the element. The three statically independent end forces, that are work conjugate to these displacements, are the axial force p_1^b and two end bending moments p_2^b and p_3^b. These element forces and displacements are grouped in vectors

$$\mathbf{p}^b = \left\langle p_1^b \quad p_2^b \quad p_3^b \right\rangle^{\mathrm{T}} \qquad \mathbf{d}^b = \left\langle d_1^b \quad d_2^b \quad d_3^b \right\rangle^{\mathrm{T}} \tag{3}$$

2.2 Kinematics

The proposed formulation is based on Timoshenko beam theory, which assumes that plane cross-sections remain plane but not necessarily perpendicular to the reference axis after deformation. It is also assumed that cross sections do not distort in their plane and that the angle of rotation of the cross-section is small, so that it can be approximated by the tangent.

The displacements quantities that describe the motion of the beam, in the basic system (simply supported beam) are grouped in the following vector

$$\mathbf{u}^b(x) = \left\langle u^b(x) \quad v^b(x) \quad \theta^b(x) \right\rangle^{\mathrm{T}} \tag{4}$$

where components $u^b(x)$ and $v^b(x)$ are the axial and transverse displacements, respectively, and $\theta^b(x)$ is the rotation of the section at a point with coordinate x, disregarding the rigid body part of the motion (i.e., these displacements are measured with respect to the chord), as shown in Figure 4.

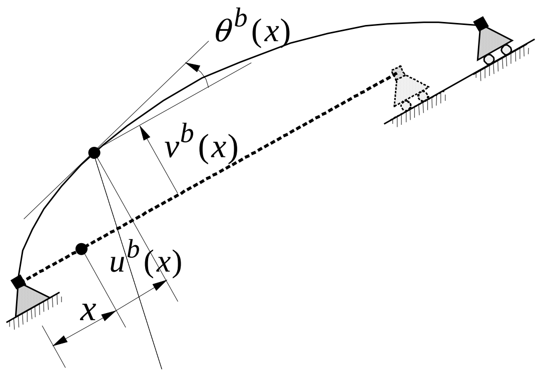

Figure 4 – Displacements of the reference axis of the beam.

According to Timoshenko beam theory, the deformations of the beam are

$$\varepsilon_0(x) = \frac{\mathrm{d}u^b(x)}{\mathrm{d}x} \qquad \gamma(x) = \frac{\mathrm{d}v^b(x)}{\mathrm{d}x} - \theta(x) \qquad \varphi(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(v^b(x)\right) \tag{5}$$

where $\varepsilon_0(x)$ is the axial deformation at the beam reference axis, $\gamma(x)$ is the shear deformation, and $\varphi(x)$ is the curvature. These deformations are grouped in the following vector

$$\mathbf{e}(x) = \left\langle \varepsilon_0(x) \quad \gamma(x) \quad \varphi(x) \right\rangle^{\mathrm{T}} \tag{6}$$

which is henceforth denoted vector of generalized section deformations.

2.3 Equilibrium equations

The equilibrium equations for a beam subjected to longitudinal and transversal distributed loads $q_x(x)$ and $q_y(x)$, respectively, are

$$\frac{\mathrm{d}N(x)}{\mathrm{d}x} + q_x(x) = 0 \qquad \frac{\mathrm{d}V(x)}{\mathrm{d}x} + q_y(x) = 0 \qquad \frac{\mathrm{d}M(x)}{\mathrm{d}x} + V(x) = 0 \tag{7}$$

where $N(x)$, $V(x)$ and $M(x)$ are the axial force, shear force and bending moment, respectively, at a section with coordinate x at the reference axis. The boundary conditions for the simply supported beam (beam in the basic system) are

$$N(L) = p_1^b \qquad M(0) = -p_2^b \qquad M(L) = p_3^b \tag{8}$$

The homogeneous solution of the differential equations (7) can be written in the form

$$\mathbf{s}(x) = \mathbf{b}(x)\mathbf{p}^b \tag{9}$$

where

$$\mathbf{s}(x) = \left\langle N(x) \quad V(x) \quad M(x) \right\rangle^{\mathrm{T}} \tag{10}$$

is the vector of section forces, and

$$\mathbf{b}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{L} & \dfrac{1}{L} \\ 0 & \dfrac{x}{L} - 1 & \dfrac{x}{L} \end{bmatrix} \tag{11}$$

is a force interpolation matrix, which is exact, regardless of the variation of the cross-section, as opposed to displacement interpolation matrices which are usually **assumed** to be formed by polynomial terms.

2.4 Section constitutive relation

The relation between section deformations and section forces is written in the form

$$\mathbf{s}(x) = \mathbf{k}_S(x)\mathbf{e}(x) \tag{12}$$

where $\mathbf{k}_S(x)$ is the stiffness matrix of the section

$$\mathbf{k}_S(x) = \begin{bmatrix} \int_A E\,\mathrm{d}A & 0 & -\int_A Ey\,\mathrm{d}A \\ 0 & GA_S & 0 \\ -\int_A Ey\,\mathrm{d}A & 0 & \int_A Ey^2\,\mathrm{d}A \end{bmatrix} \tag{13}$$

in which E is the Young modulus, G is the shear modulus, A is the area of the cross-section, and A_S is the shear area. The shear area is usually computed as

$$A_S = \kappa A \tag{14}$$

with κ being the shear coefficient, which depends on the shape of the cross-section (commonly taken equal to 5/6, for rectangular homogeneous cross-sections).

The section constitutive relation can also be written in the inverse form

$$\mathbf{e}(x) = \mathbf{f}_S(x)\mathbf{s}(x) \tag{15}$$

where $\mathbf{f}_S(x) = \mathbf{k}_S^{-1}(x)$ is the section flexibility matrix.

3. ELEMENT FORMULATION FOR STATIC ANALYSIS

This section describes the derivation of the element stiffness matrix. Although this derivation is not an original contribution, it is outlined here for the sake of completeness.

3.1 Element flexibility matrix

The relation between displacements and forces in the basic system, can be obtained based on the principle of virtual forces, and results in

$$\mathbf{d}^b = \mathbf{f}\mathbf{p}^b \tag{16}$$

where

$$\mathbf{f} = \int_0^L \mathbf{b}(x)^{\mathrm{T}} \mathbf{f}_S(x)\mathbf{b}(x)\,\mathrm{d}x \tag{17}$$

is the element flexibility matrix in the basic system, which is exact regardless of the variation of the cross-section (within the error introduced by the numerical integration procedure).

3.2 Element stiffness matrix in the basic system

The element stiffness matrix in the basic system is obtained by inversion of the flexibility matrix

$$\mathbf{k}^b(x) = \mathbf{f}^{-1}(x) \tag{18}$$

such that

$$\mathbf{p}^b = \mathbf{k}^b \mathbf{d}^b \tag{19}$$

As the element flexibility matrix is exact, the element stiffness matrix in the basic system, obtained this way, is also exact, regardless of the variation of the cross-section.

3.3 Transformation between basic and local coordinate systems

The displacements in the basic system is obtained from the local displacements with the kinematic relation

$$\mathbf{d}^b = \mathbf{T}\mathbf{d}^l \tag{20}$$

where

$$\mathbf{T} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{L} & 1 & 0 & -\frac{1}{L} & 0 \\ 0 & \frac{1}{L} & 0 & 0 & -\frac{1}{L} & 1 \end{bmatrix} \tag{21}$$

is a transformation matrix. Accordingly, forces in the local system are obtained from forces in the basic system with the relation,

$$\mathbf{p}^l = \mathbf{T}^{\mathrm{T}} \mathbf{p}^b \tag{22}$$

which can be derived from equilibrium considerations, or from the Principle of Virtual Displacements.

3.4 Element stiffness matrix in the local system

The combination of eqs. (19), (20) and (22) leads to the following expression, which relates local forces $\mathbf{p}^l$ to local displacements $\mathbf{d}^l$

$$\mathbf{p}^l = \mathbf{k}^l \mathbf{d}^l \tag{23}$$

where

$$\mathbf{k}^l = \mathbf{T}^{\mathrm{T}} \mathbf{k}^b \mathbf{T} \tag{24}$$

is the element stiffness matrix with respect to the local system

3.5 Transformation between local and global coordinate systems

Displacements in the local system can be obtained from global displacements with the usual relation

$$\mathbf{d}^l = \mathbf{R}\mathbf{d}^g \tag{25}$$

where

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

is a rotation matrix. Accordingly, forces in the global system are obtained from forces in the local system with the relation

$$\mathbf{p}^g = \mathbf{R}^{\mathrm{T}}\mathbf{p}^l \tag{27}$$

as commonly described in textbooks of structural analysis.

3.6 Element stiffness matrix in global coordinates

Combination of eqs. (23), (25) and (27) leads do the element stiffness in global coordinates

$$\mathbf{k}^g = \mathbf{R}^{\mathrm{T}}\mathbf{k}^l\mathbf{R} \tag{28}$$

such that

$$\mathbf{p}^g = \mathbf{k}^g\mathbf{d}^g \tag{29}$$

4. ELEMENT FORMULATION FOR DYNAMIC ANALYSIS

The element stiffness matrix derived in the preceding section is exact (to within the numerical accuracy of the integration scheme) regardless of the variation of the cross-section along the element length. Next, a mass matrix, consistent with the exact stiffness matrix, is derived based upon the Principle of Virtual Work.

For this purpose, exact shape functions are derived first, then the consistent mass matrix is derived starting from these shape functions.

4.1 Shape Functions in the basic system

The displacements $\mathbf{u}^b(x)$ at one point of coordinate *x* of the beam can be computed using the Principle of Virtual Forces, considering virtual loads $\delta\mathbf{p}_x = \langle \delta p_{x1} \quad \delta p_{x2} \quad \delta p_{x3} \rangle^{\mathrm{T}}$ applied at point *x*, as represented in Figure 5.

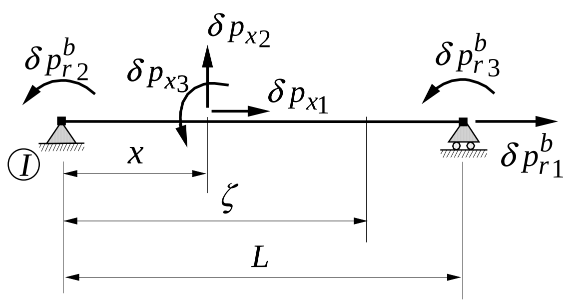

Figure 5 – Basic system with virtual forces applied at a point of coordinate *x*.

The Principle of Virtual Forces can be expressed as

$$\delta W_{\text{ext}} = \delta W_{\text{int}} \tag{30}$$

where δW_{ext} and δW_{int} are the external and internal virtual work, respectively. Eq. (30) can be written in expanded form as

$$\underbrace{\delta \mathbf{p}_r^{b\,\mathrm{T}} \mathbf{d}^b + \delta \mathbf{p}_x^{\mathrm{T}} \mathbf{u}^b(x)}_{\delta W_{\text{ext}}} = \underbrace{\int_0^L \delta \mathbf{s}^{\mathrm{T}}(\zeta, x)\mathbf{e}(\zeta)\,\mathrm{d}\zeta}_{\delta W_{\text{int}}} \tag{31}$$

where $\delta \mathbf{s}(\zeta, x)$ are the virtual internal forces at a point of coordinate ζ due to virtual loads $\delta \mathbf{p}_x$ applied at a point of coordinate *x*, and $\delta \mathbf{p}_r^b = \left\langle \delta p_{r1}^b \quad \delta p_{r2}^b \quad \delta p_{r3}^b \right\rangle^{\mathrm{T}}$ are the virtual forces at the element ends (in equilibrium with the applied virtual forces $\delta \mathbf{p}_x$), as indicated in Figure 5.

Any system of virtual forces in equilibrium, not necessarily satisfying compatibility of displacements, can be chosen as an admissible system for this purpose.

Theoretically, all admissible systems should provide the same displacements $\mathbf{u}^b(x)$. However, due to numerical integration error, the results are usually different. In this paper, the system shown in Figure 6 is employed. The system is obtained by “cutting” the beam into two parts, at the point of application of the virtual loads. Then, each part of the beam is considered as a cantilever, supporting half of the virtual loads at the free ends, as illustrated in the figure. This system was chosen because it is invariant with respect to node numbering (i.e., if element nodes *I* and *J* are swapped, the results remain unchanged, regardless of the errors introduced by the numerical integration scheme). This is an advantage with respect to the formulation proposed by Molins *et al*. (1998), which is not invariant with respect to node numbering.

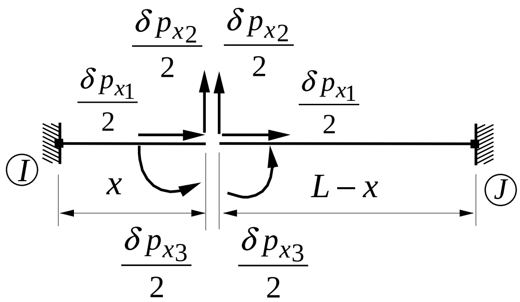

Figure 6 – Primary system with virtual loads (beam split into two parts).

A system of virtual forces in equilibrium is easily determined by computing the support reactions of the two cantilevers. The results are shown in Figure 7.

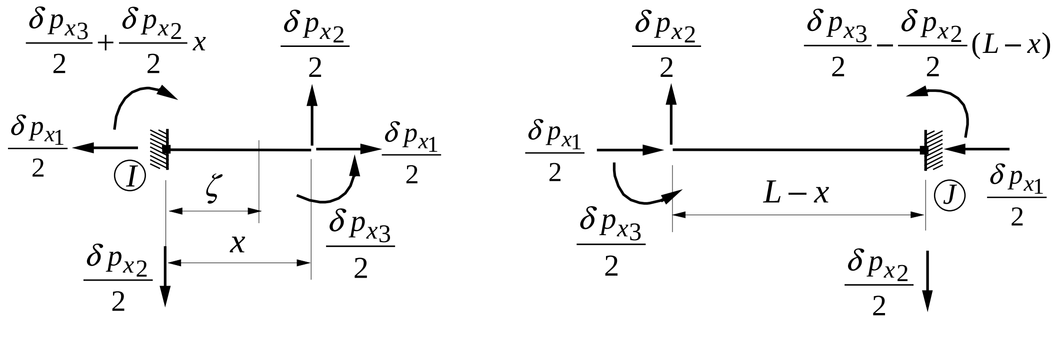

Figure 7 – Primary system with admissible virtual forces (forces in equilibrium).

The virtual internal forces $\delta\mathbf{s}(\zeta, x)$, for the primary system shown in Figure 7 can be computed as

$$\delta\mathbf{s}(\zeta, x) = \mathbf{b}_v(\zeta, x)\,\delta\mathbf{p}_x(x) \tag{32}$$

where

$$\mathbf{b}_v(\zeta, x) = \begin{cases} \dfrac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & x-\zeta & 1 \end{bmatrix} & \text{if } \zeta < x \\ -\dfrac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & x-\zeta & 1 \end{bmatrix} & \text{if } \zeta > x \end{cases} \tag{33}$$

which can be easily confirmed by computing the virtual internal force diagrams, for both cantilevers.

For the system shown in Figure 5, the virtual forces $\delta\mathbf{p}_r^b$ at the element ends are

$$\delta\mathbf{p}_r^b = \mathbf{b}_r(x)\delta\mathbf{p}_x \tag{34}$$

where

$$\mathbf{b}_r(x) = -\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 1 \\ 0 & x-L & 1 \end{bmatrix} \tag{35}$$

Substitution of eqs. (32) and (34) into eq. (31) leads to

$$\delta\mathbf{p}_x{}^{\mathrm{T}}\mathbf{b}_r(x)^{\mathrm{T}}\mathbf{d}^b + \delta\mathbf{p}_x{}^{\mathrm{T}}\mathbf{u}^b(x) = \delta\mathbf{p}_x{}^{\mathrm{T}}\int_0^L \mathbf{b}_v(\zeta, x)^{\mathrm{T}}\mathbf{e}(\zeta)\,\mathrm{d}\zeta \tag{36}$$

As the virtual forces $\delta\mathbf{p}_x$ are arbitrary, eq. (36) can be rewritten as

$$\mathbf{u}^b(x) = \int_0^L \mathbf{b}_v(\zeta, x)^{\mathrm{T}}\mathbf{e}(\zeta)\,\mathrm{d}\zeta - \mathbf{b}_r(x)^{\mathrm{T}}\mathbf{d}^b \tag{37}$$

Substitution of eq. (19) into eq. (9), and the result into eq. (15) allows for the computation of the section deformations in terms of the end displacements

$$\mathbf{e}(\zeta) = \mathbf{f}_s(\zeta)\mathbf{s}(\zeta) = \mathbf{f}_s(\zeta)\mathbf{b}(\zeta)\mathbf{p}^b = \mathbf{f}_s(\zeta)\mathbf{b}(\zeta)\mathbf{f}^{-1}\mathbf{d}^b \tag{38}$$

Substitutition of eq. (38) into eq. (37) gives

$$\mathbf{u}^b(x) = \mathbf{N}^b(x)\mathbf{d}^b \tag{39}$$

where

$$\mathbf{N}^b(x) = \int_0^L \mathbf{b}_v(\zeta, x)^{\mathrm{T}}\mathbf{f}_s(\zeta)\mathbf{b}(\zeta)\,\mathrm{d}\zeta\,\mathbf{f}^{-1} - \mathbf{b}_r(x)^{\mathrm{T}} \tag{40}$$

is a matrix that allows for the computation of the basic displacement field $\mathbf{u}^b(x)$ in terms of basic end displacements $\mathbf{d}^b$. It should be noted that this is, in fact, a matrix of exact shape functions for the beam in the basic system, as there is no approximate interpolation involved in the presented derivations.

4.2 Sectional mass matrix

The inertia forces $\mathbf{p}_I^b(x)$ in basic coordinates can be computed in terms of the basic accelerations $\ddot{\mathbf{u}}^b(x)$ as

$$\mathbf{p}_I^b(x) = \mathbf{m}_s(x)\ddot{\mathbf{u}}^b(x) \tag{41}$$

where

$$\mathbf{m}_s(x) = \begin{bmatrix} \int_A \rho \,\mathrm{d}A & 0 & -\int_A \rho y \,\mathrm{d}A \\ 0 & \int_A \rho \,\mathrm{d}A & 0 \\ -\int_A \rho y \,\mathrm{d}A & 0 & \int_A \rho y^2 \,\mathrm{d}A \end{bmatrix} \tag{42}$$

with ρ being the mass density of the material.

4.3 Element mass matrix in the basic system

A mass matrix with reference to the basic coordinate system can be determined using the Principle of Virtual Displacements, which results in the following expression

$$\mathbf{m}^b = \int_0^L \mathbf{N}^b(x)^{\mathrm{T}} \mathbf{m}_s(x) \mathbf{N}^b(x) \,\mathrm{d}x \tag{43}$$

4.4 Shape functions in the local coordinate system

As a counterpart for the displacement fields $\mathbf{u}^b(x)$ of the beam in the basic system (see eq. (4)), displacement fields can also be defined with respect to the local system, according to Figure 8, as

$$\mathbf{u}^l(x) = \left\langle u^l(x) \quad v^l(x) \quad \theta^l(x) \right\rangle^{\mathrm{T}} \tag{44}$$

The displacement vector $\mathbf{u}^l(x)$ in the local coordinate system can be obtained from the basic displacements $\mathbf{u}^b(x)$, according to Figure 8. Considering that the angle α, between the basic and local systems is small, the following linear geometric relation can be obtained

$$\mathbf{u}^l(x) = \mathbf{u}^b(x) + \mathbf{N}_{\mathrm{rig}}^l(x)\mathbf{d}^l \tag{45}$$

in which

$$\mathbf{N}_{\mathrm{rig}}^l(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{x}{L} & 0 & 0 & \frac{x}{L} & 0 \\ 0 & -\frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 \end{bmatrix} \tag{46}$$

is an interpolation matrix corresponding to rigid body displacements only, i.e., it excludes the deformation modes from an arbitrary displacement vector $\mathbf{d}^l$.

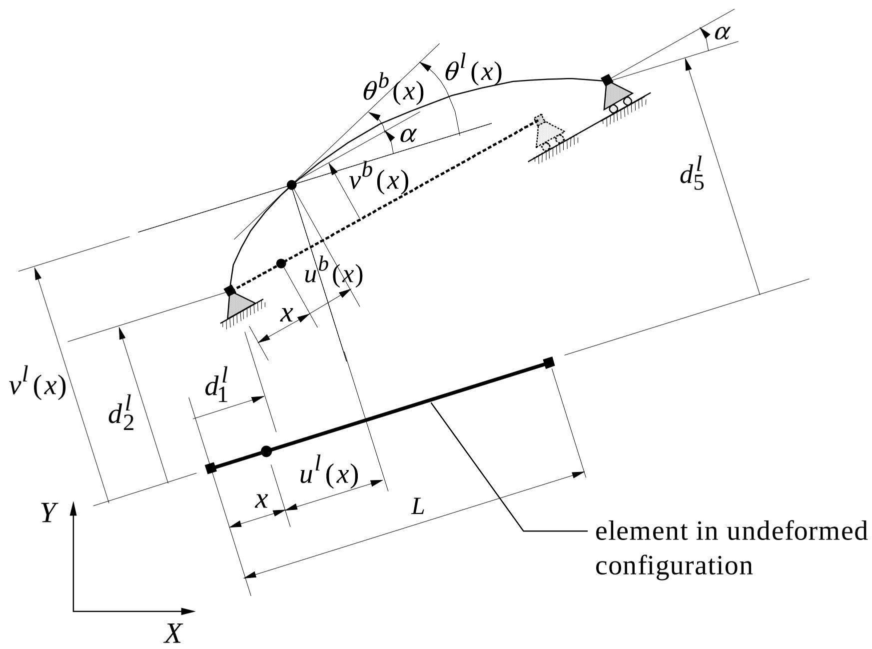

Figure 8 – Displacement fields in basic and local coordinates.

Substitution of eq. (20) into eq. (39) and the result into eq. (45) yields

$$\mathbf{u}^l(x) = \left[\mathbf{N}^b(x)\mathbf{T} + \mathbf{N}^l_{\mathrm{rig}}(x)\right]\mathbf{d}^l = \mathbf{N}^l(x)\mathbf{d}^l \tag{47}$$

in which,

$$\mathbf{N}^l(x) = \mathbf{N}^b(x)\mathbf{T} + \mathbf{N}^l_{\mathrm{rig}}(x) = \mathbf{N}^l_{\mathrm{def}}(x) + \mathbf{N}^l_{\mathrm{rig}}(x) \tag{48}$$

is the displacement interpolation matrix in local coordinates (shape functions), with

$$\mathbf{N}^l_{\mathrm{def}}(x) = \mathbf{N}^b(x)\mathbf{T} \tag{49}$$

Thus, it can be concluded that the shape functions in the local coordinate system is composed of two terms: $\mathbf{N}^l_{\mathrm{def}}(x)$, which corresponds to deformation modes; $\mathbf{N}^l_{\mathrm{rig}}(x)$, which corresponds to the rigid part of the motion;

4.5 Element mass matrix in the local coordinate system

The mass matrix in local coordinates is obtained with the Principle of Virtual Displacements, and results in the following expression

$$\begin{aligned}\mathbf{m}^l &= \int_0^L \mathbf{N}^l(x)^{\mathrm{T}} \mathbf{m}_s(x) \mathbf{N}^l(x)\,\mathrm{d}x = \int_0^L \left(\mathbf{N}^l_{\mathrm{def}}(x) + \mathbf{N}^l_{\mathrm{rig}}(x)\right)^{\mathrm{T}} \mathbf{m}_s(x) \left(\mathbf{N}^l_{\mathrm{def}}(x) + \mathbf{N}^l_{\mathrm{rig}}(x)\right)\mathrm{d}x \\ &= \int_0^L \left(\mathbf{N}^{l\,\mathrm{T}}_{\mathrm{def}} \mathbf{m}_s \mathbf{N}^l_{\mathrm{def}} + \mathbf{N}^{l\,\mathrm{T}}_{\mathrm{def}} \mathbf{m}_s \mathbf{N}^l_{\mathrm{rig}} + \mathbf{N}^{l\,\mathrm{T}}_{\mathrm{rig}} \mathbf{m}_s \mathbf{N}^l_{\mathrm{def}} + \mathbf{N}^{l\,\mathrm{T}}_{\mathrm{rig}} \mathbf{m}_s \mathbf{N}^l_{\mathrm{rig}}\right)\mathrm{d}x\end{aligned} \tag{50}$$

where the argument (x) has been omitted in the last equality for brevity of notation.

Substitution of eq. (49) into eq. (50), and consideration of eq. (43), leads to

$$\mathbf{m}^l = \mathbf{T}^{\mathrm{T}}\mathbf{m}^b\mathbf{T} + \mathbf{m}^l_{\mathrm{rig}} \tag{51}$$

in which

$$\mathbf{m}^l_{\text{rig}} = \int_0^L \left(\mathbf{N}^{l\ \mathrm{T}}_{\text{def}} \mathbf{m}_s \mathbf{N}^l_{\text{rig}} + \mathbf{N}^{l\ \mathrm{T}}_{\text{rig}} \mathbf{m}_s \mathbf{N}^l_{\text{def}} + \mathbf{N}^{l\ \mathrm{T}}_{\text{rig}} \mathbf{m}_s \mathbf{N}^l_{\text{rig}} \right) \mathrm{d}x \tag{52}$$

is a term of the mass matrix corresponding to rigid body mode effects. Thus, it has been shown that in contrast with the stiffness matrix which transforms according to eq. (24), the mass matrix transforms according to eq. (51).

The additional term $\mathbf{m}^l_{\text{rig}}$ is present in eq. (51) because, as opposed to the stiffness matrix $\mathbf{k}^l$, which has rank 3 due to the presence of rigid body modes, the mass matrix $\mathbf{m}^l$ has full rank (6), and thus cannot be transformed directly from matrix $\mathbf{m}^b$, which is a 3×3 matrix.

Regarding numerical integration, it can be observed from eqs. (40), (48) and (50), that the expression for the mass matrix involves the computation of a double integral. This is an advantage with respect to the formulation proposed by Molins *et al*. which requires evaluation of a triple integral for the computation of the mass matrix.

4.6 Element mass matrix in the global coordinate system

The mass matrix in the global coordinate system is computed simply by

$$\mathbf{m}^g = \mathbf{R}^{\mathrm{T}} \mathbf{m}^l \mathbf{R} \tag{53}$$

as usually done in most finite element codes.

5. NUMERICAL EXAMPLES

The proposed formulation has been implemented in a general finite element code, written in MatLab®. The program efficiency has been verified using some numerical examples.

The solutions obtained with the present formulation are compared with those obtained with other standard elements.

5.1 Validation Tests

In this section will be presented tests with the goal to validate the finite element proposed in this paper.

Cantilever beam with varying cross-section. The first validation test is a cantilever beam with rectangular section varying linearly along the element axis, as illustrated in Figure 9. The beam has length $L = 5\,\text{m}$, Young's modulus $E = 10^6\,\text{kN/m}^2$, Poisson's ratio $\nu = 0.3$, and mass density $\rho = 1\,\text{kg/m}^3$. The section dimensions are: initial depth $h_I = 1\,\text{m}$; initial width $b_I = 1\,\text{m}$; final depth $h_J = 0.3\,\text{m}$; final width $b_J = 0.3\,\text{m}$.

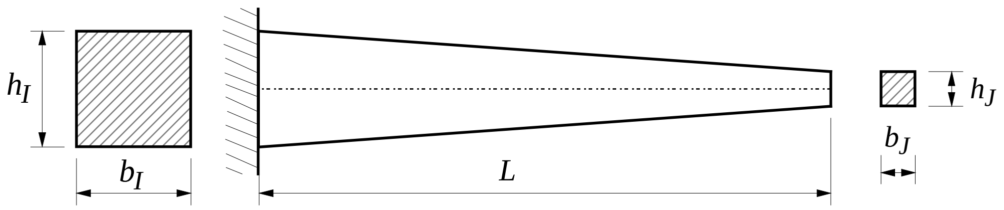

Figure 9 – Example 1: straight cantilever beam with varying cross-section.

The natural frequencies of the first three vibration modes are evaluated for this structure. These frequencies are computed varying the order of numerical integration from 4 to 10 Gauss points, but considering just one finite element to represent the entire beam. The results are compared against the results obtained with the element proposed by Molins *et al*. (1998) in Figure 10. The analyses performed with the formulation proposed by Molins *et al*. consider two different schemes to number the element nodes: in the first scheme, denoted *IJ* in the figure, node *I* corresponds to the node at the left-hand side, and node *J* corresponds to the node at the right-hand side of the element; in the second scheme, denoted *JI*, the connectivity is reversed.

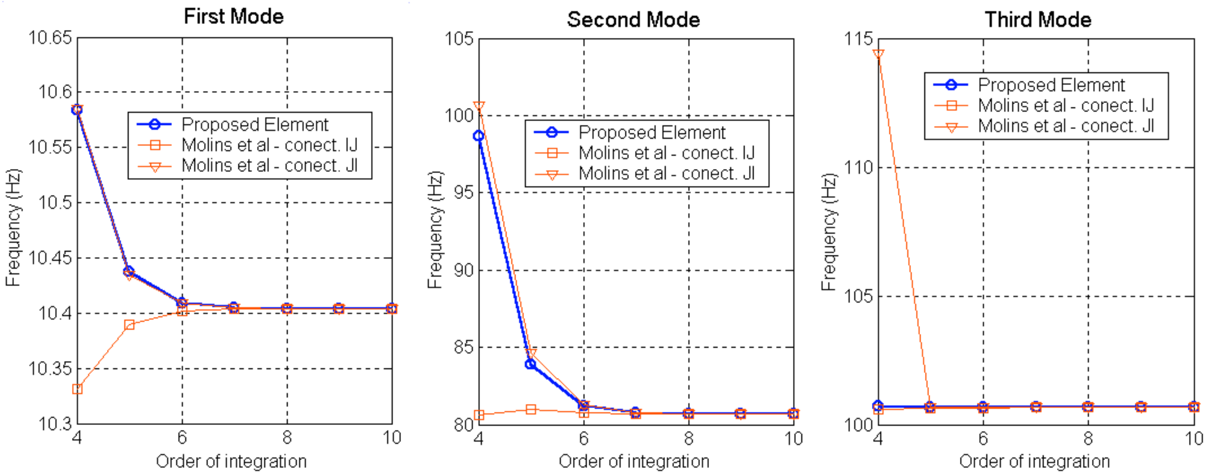

Figure 10 – Example 1: Convergence analysis - Frequency against Order of Integration.

The mass matrix of the beam, obtained with four Gauss integration points, and considering just one element to represent the entire beam, is shown in Figure 11. The results obtained with the present formulation and with the formulation proposed by Molins *et al*. (1998) are shown, for the two different connectivity schemes (*IJ* and *JI*). It may be noted that when connectivity is reversed, the 3-by-3 submatrices (shown in the same collors) obtained with the proposed element are identical, while those obtained by the formulation proposed by Molins *et al*. are different, due to error introduced by the numerical integration procedure.

Proposed Element – Conect. IJ

m =

1.6672	0	0	0.2498	0	0
0	1.7879	1.8555	0	0.1700	-0.0633
0	1.8555	2.7666	0	0.3781	-0.1630
0.2498	0	0	0.1499	0	0
0	0.1700	0.3781	0	0.1888	-0.0685
0	-0.0633	-0.1630	0	-0.0685	0.0282

Proposed Element – Conect. JI

m =

0.1499	0	0	0.2498	0	0
0	0.1888	-0.0685	0	0.1700	0.3781
0	-0.0685	0.0282	0	-0.0633	-0.1630
0.2498	0	0	1.6672	0	0
0	0.1700	-0.0633	0	1.7879	1.8555
0	0.3781	-0.1630	0	1.8555	2.7666

Molins et al Element – Conect. IJ

m =

1.6664	0	0	0.2500	0	0
0	1.7024	1.6441	0	0.2035	-0.1170
0	1.6441	2.3610	0	0.4002	-0.2227
0.2501	0	0	0.1502	0	0
0	0.2046	0.4040	0	0.2062	-0.0839
0	-0.1171	-0.2233	0	-0.0853	0.0394

Molins et al Element – Conect. JI

m =

0.1500	0	0	0.2503	0	0
0	0.1888	-0.0602	0	0.2220	0.4676
0	-0.0758	0.0266	0	-0.1266	-0.2577
0.2492	0	0	1.6672	0	0
0	0.1191	-0.0026	0	1.7868	1.9516
0	0.2933	-0.0781	0	1.7509	2.7505

Figure 11 – Mass matrices of cantilever beam of example 1, and comparison regarding element connectivity.

As discussed previously, the formulation proposed by Molins *et al* (1998) is not invariant with respect to element node numbering. In other words, if the element connectivity is reversed from *IJ* to *JI* the results change, due to the approximation in the numerical integration procedure. This can be confirmed with the results shown in the plots (Figure 10). The error is considerable for a low integration order, and tends do vanish as the number of integration points increases. The formulation proposed in this paper does not have this shortcoming; i.e., it is invariant with respect to element connectivity.

Despite this discrepancy, it is observed from the plots that the present formulation and the formulation proposed by Molins *et al*. (1998) converge to the same values as the order of integration increases.

Simply supported beam with varying cross-section. Another test is carried out considering a simply supported beam with the same geometry and material properties of the previous example, as shown in Figure 12. The beam is subdivided into smaller elements in order to verify the convergence properties of the proposed formulation. The discretization consists of 1, 2, 4, 8, 16 and 32 elements. The results of the natural frequencies for the first and second vibration modes are presented in Figure 13. The order of integration used was 5.

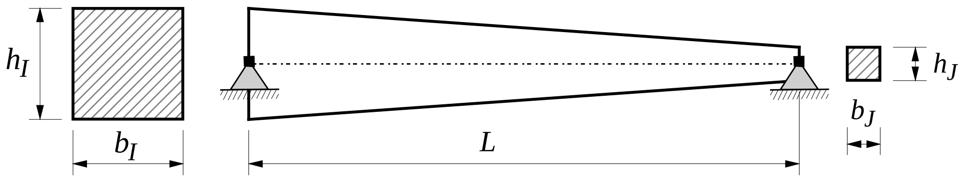

Figure 12 – Example 2: simply supported beam with varying cross-section.

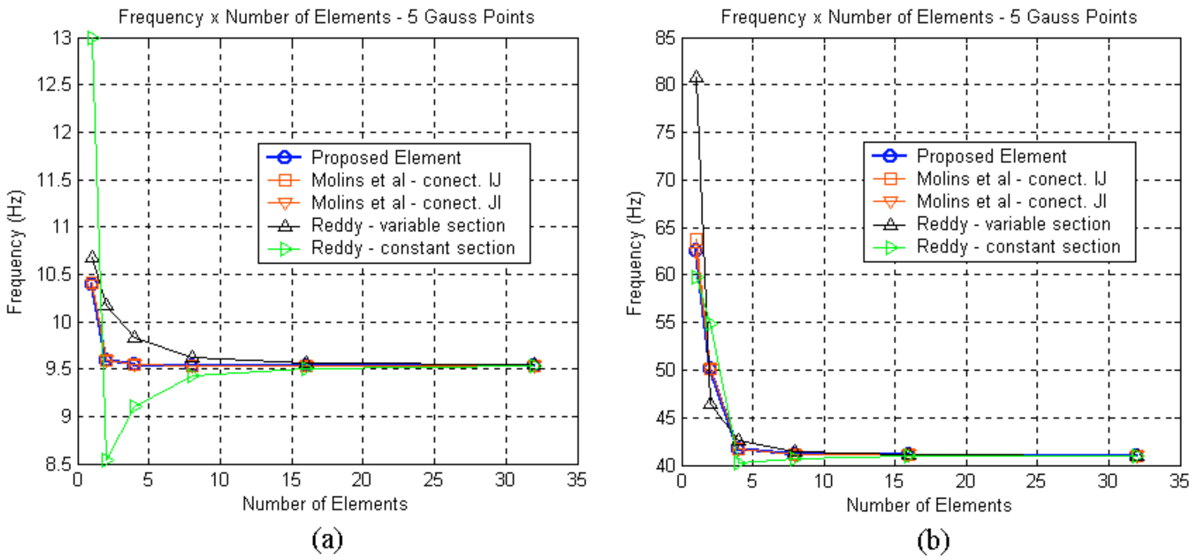

Figure 13 – Example 2: (a) First natural frequency; (b) Second natural frequency

Figure 13 shows the results obtained by the proposed formulation, and by the formulations proposed by Molins *et al* (1998), and by Reddy (1997). The formulation proposed by Reddy (1997) presents a locking-free displacement finite element model using the form of the exact solution of the Timoshenko beam theory for prismatic beams.

The results demonstrate a better convergence rate for the flexibility based elements.

Portal frame with non-prismatic beams. The last validation test consists of a symmetric portal frame as illustrated in Figure 14. The indicated dimensions are: $L_1 = 10.0\text{m}$, $L_2 = 3.5\text{m}$, $L_3 = 3.0\text{m}$, $h_1 = 0.4\text{m}$, $h_2 = 1.42\text{m}$, $h_3 = 0.4\text{m}$. The base width of all sections is $b = 2.0\text{m}$. The material properties are: Young's modulus $E = 25\times10^6 \text{kN/m}^2$, Poisson's ratio $\nu = 0.3$, and specific weight $\gamma = 25\,\text{kN}/\text{m}^3$. The integration order used was equal to eight.

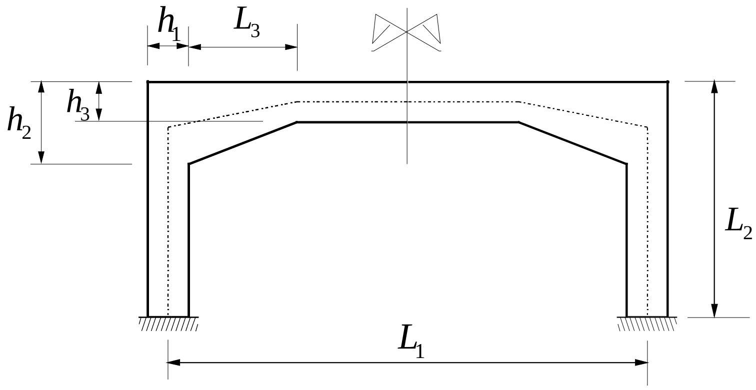

Figure 14 – Example 3: (a) Structure; (b) model discretization by FEM.

The structure was analyzed considering three different levels of discretization, with a total of five, ten and twenty elements (which correspond to one, two and three elements per structural member, respectively). The results for the different studied elements are shown in Figure 15.

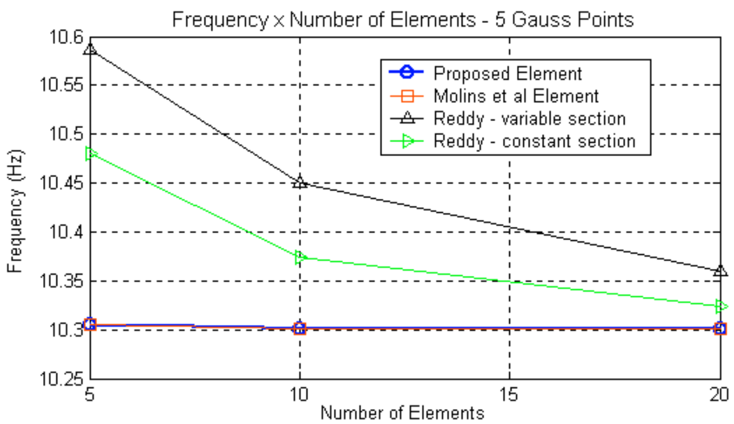

Figure 15 – Example 3: Convergence analysis - Frequency against number of elements.

From the results it is observed that the flexibility-based elements present a better convergence rate.

6. SUMMARY AND CONCLUSIONS

A new expression for the computation of the mass matrix of a Timoshenko beam finite element of varying cross-section has been presented in this paper. The formulation developed in the paper is based on exact force interpolation functions, which allows for the computation of an exact stiffness matrix, and its corresponding consistent mass matrix (i.e., a mass matrix obtained from exact shape functions).

This was accomplished by determining the exact element flexibility matrix in a system without rigid body modes, denoted basic system. The exact element stiffness matrix in the basic system is then obtained computing the inverse of the flexibility matrix.

The paper also demonstrated that exact shape functions in the basic system could be determined for a beam of varying cross-section based on the Principle of Virtual forces. The adopted primary system of virtual forces in equilibrium corresponded to the original beam element divided into two cantilever beams, which rendered the formulation invariant to element connectivity, despite the presence of numerical integration errors.

From the exact shape functions in the basic system, a consistent mass matrix could be computed in this auxiliar system. However, it has been shown that although the local stiffness matrix can be obtained from the basic stiffness matrix, through a usual matrix transformation equation ($\mathbf{k}^l = \mathbf{T}^{\mathrm{T}}\mathbf{k}^b\mathbf{T}$), the mass matrix cannot be transformed in the same way because it has full rank, as opposed to the stiffness matrix, which has rank 3. Nevertheless, the consistent mass matrix in the local coordinate system can be computed using the exact expressions for the shape functions in the local system, which were transformed from the basic system using a simple kinematic transformation.

Regarding numerical integration, it was observed that the proposed expression for the mass matrix involved the computation of a double integral. Nonetheless, this is an advantage with respect to the formulation proposed by Molins *et al*, which requires evaluation of a triple integral for the computation of the mass matrix.

Some numerical examples have been analyzed using the developed formulation, and the results demonstrated the accuracy and efficiency of the proposed element. The test cases illustrated the applicability of the proposed formulation for solving dynamic problems.

So far, the proposed method has only been applied to straight planar beams, but the authors believe that this formulation can be extended, without much difficulty, to curved spatial elements.

Acknowledgements

Part of this work was carried out under the financial support of a scholarship awarded by the Brazilian Ministry of Education, through its agency CAPES, to the first author, during his Ph.D. at UC Berkeley. The first author gratefully acknowledges this support.

The third author wishes to acknowledge the PARD program of the Federal University of Pará, for his research assistantship during his B.Sc., as a financial support for this study.

REFERENCES

Backlund, J., 1974, Large deflection analysis of elasto-plastic beams and frames. Int. J. Mech. Sci., 18, 269-277.

Carol, I., and Murcia, J. 1989, Nonlinear time-dependent analysis of planar frames using an 'exact' formulation – I: Theory. Comp. and Struct., 33, 79-87.

Chen, C.-N., 2000, Dynamic equilibrium equations of non-prismatic beams defined on an arbitrarily selected co-ordinate system, Journal of Sound and Vibration, vol. 230, pp. 241-260.

Chen, C.-N., 2002, DQEM vibration analyses of non-prismatic shear deformable beams resting on elastic foundations, Journal of Sound and Vibration, vol. 255, pp. 989-999.

Chopra, A. K., 1995, Dynamics of Structures, Prentice Hall.
Clough, R. W. , Penzien, J., 1993, Dynamics of Structures, McGraw Hill.
De Souza, R. M., 1999, “Force-based Finite Element for Large Displacement Inelastic Analysis of Frames”, Ph.D. Dissertation, Department of Civil and Environmental Engineering, University of California, Berkeley, USA.
Gajewski, A., 2001, Vibration and stability of a non-prismatic column compressed by non-conservative forces in non-linear creep conditions, Journal of Sound and Vibration, vol. 248, pp. 315-327.
Molins, C., Roca, P. and Barbat, A. H., 1998, Flexibility-based linear dynamic analysis of complex structures with curved-3d beams, Earthquake Engineering Structural Dynamics, vol. 27, pp. 731-747.
Neuenhofer, A., and Filippou, F. C. 1997, Evaluation of nonlinear frame finite-element models. J. Struct. Engrg., ASCE, 123, 958-966.
Reddy, J. N., 1997, On locking-free shear deformable beam finite elements, Computer Methods Appl. Mech. Engrg., vol. 149, pp. 113-132.
Ruta, P., 2002, Dynamic stability problem of a non-prismatic rod, Journal of Sound and Vibration, vol. 250, pp. 445-464.
Spacone, E., Ciampi, V., and Filippou, F. C. 1996, “Mixed formulation of nonlinear beam finite element, Comp. and Struct., 58, 71-83.